Theory Lunch Talk

Connection between Permanents and product of linear forms.

$\mathrm{Per}(A) = \sum_{\sigma \in S^n} \prod_{i=1}^{n} A_{i\sigma(i)}$, #P-hard, NP-~~complete~~ hard to determine sign.

If $A \succeq B \succeq 0$, $\mathrm{Per}(A) \geq \mathrm{Per}(B) \geq 0$.

Anari et. al. : c^n - multiplicative approx., where $c = e^{1+\gamma} \approx 4.84$

Also showed : α^n improvement not possible*, for any $\alpha > 1$

Our contribution;

- Connection to opt. problem (simplify proof)
- improve approx factor by $e^{O(\sqrt{n})}$
- Van-der Waerden's conj for PSD matrices.

Let $A = V^\dagger V$ V has columns v_i.

$$r(A) = \max_{\substack{\|x\| = \sqrt{n} \\ x \in \mathbb{C}^n}} \prod_{i=1}^{n} |\langle x, v_i \rangle|^2$$

① relaxation + rounding alg for $r(A)$

$$e^{-nL_r} \mathrm{rel}(A) \overset{(d)}{\leq} r(A) \overset{(a)}{\leq} \mathrm{rel}(A)$$

r is rank of relaxation solution $= \mathrm{rank}(X^*)$

② Connection to permanent:

$$e^{-n(1+L_r)} \mathrm{rel}(A) \overset{(d)}{\leq} e^{-n} r(A) \overset{(c)}{\leq} \mathrm{per}(A) \overset{(b)}{\leq} \mathrm{rel}(A)$$

(a) $\mathrm{rel}(A) =$

(CP) : $\max \prod_{i=1}^{n} v_i^\dagger X v_i$

s.t. $\mathrm{Tr}(X) = n$

$X \succeq 0$ X Hermitian

(b):

(D): $\min \lambda^n$

s.t. $\sum_{i=1}^{n} \alpha_i v_i v_i^\dagger \preceq \lambda I$

$\prod_{i=1}^{n} \alpha_i \geq 1$ $\alpha_i \geq 0$

(D*): $\min \prod_i D_i = \mathrm{per}(D)$

s.t. $A \preceq D$, d diagonal

(c): if $\|x\|^2 = n$, $xx^\dagger \preceq nI$

$V^\dagger x x^\dagger V \preceq n V^\dagger V = nA$

$$n! \prod_{i=1}^{n} |\langle x, v_i \rangle|^2 = \mathrm{per}(V^\dagger x x^\dagger V) \leq n^n \mathrm{per}(A)$$

$$\frac{n!}{n^n} r(A) \leq \mathrm{per}(A)$$

(d): $L_r = \sum_{i=1}^{r-1} \frac{1}{i} - \log(r)$

$\lim_{r\to\infty} L_r = \gamma$, $r = O(\sqrt{n})$

rounding: given $X^* = UU^\dagger$, $U \in \mathbb{R}^{n\times r}$, return feasible solution to opt. problem in $r(A)$

1. sample $z \sim \mathbb{C}N(0, I_r)$
2. return $y = \frac{\sqrt{n}\, Uz}{\|Uz\|}$

$$\mathbb{E}_z \left[\prod_{i=1}^{n} |\langle y, v_i \rangle|^2 \right] = \mathbb{E}\left[\prod_{i=1}^{n} \frac{n |\langle Uz, v_i \rangle|^2}{\|Uz\|^2} \right]$$

$$\overset{(*)}{\geq} \exp\Big(\sum_{i=1}^{n} \big(\log n + \overset{①}{\mathbb{E} \log |\langle Uz, v_i \rangle|^2} - \overset{②}{\mathbb{E} \log \|Uz\|^2} \big) \Big)$$

①: $\mathbb{E}[\log |\langle Uz, v_i \rangle|^2] = \mathbb{E}\left[\log |\langle \frac{U^\dagger v_i}{\|U^\dagger v_i\|}, z \rangle|^2 \right]$

$+ \log \|U^\dagger v_i\|^2$

$= \mathbb{E}[\log |z_1|^2] + \log v_i^\dagger U U^\dagger v_i$

$= -\gamma + \log v_i^\dagger X^* v_i$

②: $\sum_i \lambda_i = n$

$\mathbb{E}[\log z^\dagger U^\dagger U z] = \mathbb{E}[\log \sum_{i=1}^{r} \lambda_i |z_i|^2]$

$\leq \mathbb{E}[\log \frac{n}{r} \sum_{i=1}^{r} |z_i|^2]$

$= \log n + L_r - \gamma$

$$(*) \geq \exp\Big(\sum_{i=1}^{n} (\log v_i^\dagger X^* v_i - L_r) \Big)$$

$$= e^{-nL_r} \prod_{i=1}^{n} v_i^\dagger X^* v_i = e^{-nL_r} \mathrm{rel}(A)$$

Conjecture:

$$\frac{n!}{n^n} r(A) \overset{(c)}{\leq} \mathrm{per}(A) \overset{(?)}{\leq} r(A)$$

(?) implied by Pate's conjecture.

Van-der-Waerden's conj :

if A doubly stochastic,

$$\frac{n!}{n^n} \leq \mathrm{per}(A) \leq 1$$
