Rounding semidefinite relaxations of concave functions of quadratic forms

Chenyang Yuan

Based on joint work with Pablo Parrilo

Toyota Research Institute

Friday 26th July, 2024

Introduction

How to find approximations to 3 seemingly unrelated problems:

- Subspace approximation
- Max-Cut
- Permanent of PSD matrices

Introduction

How to find approximations to 3 seemingly unrelated problems:

- Subspace approximation
- Max-Cut
- Permanent of PSD matrices

Goals:

- A unified framework to encompass and generalize these problems
- Simple rounding analysis to obtain optimal approximation factors matching lower-bounds

Introduction

How to find approximations to 3 seemingly unrelated problems:

- Subspace approximation
- Max-Cut
- Permanent of PSD matrices

Goals:

- A unified framework to encompass and generalize these problems
- Simple rounding analysis to obtain optimal approximation factors matching lower-bounds

Bounding expectations of functions of Gaussians gives analysis for rounding algorithms for optimization problems on the sphere

Subspace Approximation

Given $a_1, \ldots, a_m \in \mathbb{R}^n$ and $p \geq 1$. Find $(n-d)$-dimensional subspace V that minimizes

$$\left(\sum_{i=1}^{m} \text{dist}(a_i, V)^p \right)^{\frac{1}{p}}$$

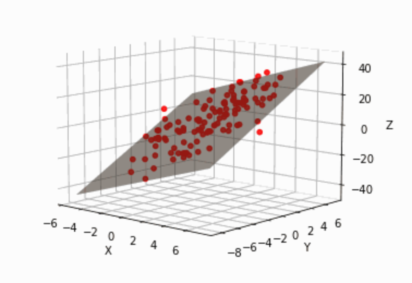
Subspace Approximation

Given $a_1, \ldots, a_m \in \mathbb{R}^n$ and $p \geq 1$. Find $(n-d)$-dimensional subspace V that minimizes

$$\left(\sum_{i=1}^{m} \text{dist}(a_i, V)^p \right)^{\frac{1}{p}}$$

$p = 2$ equivalent to least squares, polynomial-time solution via SVD.

$p = \infty$, minimize maximum distance to V, NP-hard problem in computational convex geometry.



Subspace Approximation SDP Relaxation

$$\min_V \left(\sum_{i=1}^m \mathrm{dist}(a_i, V)^p \right)^{\frac{1}{p}} = \begin{array}{ll} \min_X & \left(\sum_{i=1}^m (a_i^\top XX^\top a_i)^{p/2} \right)^{\frac{1}{p}} \\ \text{s.t.} & X^\top X = I_d \\ & X \in \mathbb{R}^{n\times d} \end{array}$$

[Gur+16] Guruswami et al. "Bypassing UGC from some optimal geometric inapproximability results". 2016.

[DTV11] Deshpande, Tulsiani, and Vishnoi. "Algorithms and hardness for subspace approximation". 2011.

Subspace Approximation SDP Relaxation

$$\min_V \left(\sum_{i=1}^m \mathrm{dist}(a_i, V)^p \right)^{\frac{1}{p}} = \begin{array}{ll} \min\limits_X & \left(\sum\limits_{i=1}^m (a_i^\top X X^\top a_i)^{p/2} \right)^{\frac{1}{p}} \\ \text{s.t.} & X^\top X = I_d \\ & X \in \mathbb{R}^{n \times d} \end{array}$$

SDP relaxation via convex hull of feasible set (Fantope):

$$\begin{array}{ll} \min\limits_Q & \left(\sum\limits_{k=1}^m (a_i^\top Q a_i)^{p/2} \right)^{\frac{1}{p}} \\ \text{s.t.} & \mathrm{Tr}(Q) = d \\ & 0 \preceq Q \preceq I_n \end{array}$$

[Gur+16] Guruswami et al. "Bypassing UGC from some optimal geometric inapproximability results". 2016.

[DTV11] Deshpande, Tulsiani, and Vishnoi. "Algorithms and hardness for subspace approximation". 2011.

Subspace Approximation SDP Relaxation

$$\min_V \left(\sum_{i=1}^m \mathrm{dist}(a_i, V)^p\right)^{\frac{1}{p}} = \begin{array}{cl} \min\limits_X & \left(\sum\limits_{i=1}^m (a_i^\top XX^\top a_i)^{p/2}\right)^{\frac{1}{p}} \\ \text{s.t.} & X^\top X = I_d \\ & X \in \mathbb{R}^{n\times d} \end{array}$$

SDP relaxation via convex hull of feasible set (Fantope):

$$\begin{array}{cl} \min\limits_Q & \left(\sum\limits_{k=1}^m (a_i^\top Q a_i)^{p/2}\right)^{\frac{1}{p}} \\ \text{s.t.} & \mathrm{Tr}(Q) = d \\ & 0 \preceq Q \preceq I_n \end{array}$$

Constant-factor approximation algorithm [DTV11] for $p \geq 1$, $0 \leq d \leq n$.

For $d = 1$ and $p > 2$, [Gur+16] shows that it is NP-hard to improve constant.

[Gur+16] Guruswami et al. "Bypassing UGC from some optimal geometric inapproximability results". 2016.

[DTV11] Deshpande, Tulsiani, and Vishnoi. "Algorithms and hardness for subspace approximation". 2011.

Max Cut

Given $A \succ 0$, the $\textsc{MaxCut}$ problem is:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle$$

[BRS15] Briët, Regev, and Saket. "Tight hardness of the non-commutative Grothendieck problem". 2015.

[Nes98] Nesterov. "Semidefinite relaxation and nonconvex quadratic optimization". 1998.

[GW95] Goemans and Williamson. "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming". 1995.

Max Cut

Given $A \succ 0$, the $\textsc{MaxCut}$ problem is:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle$$

For PSD A, we have a $\frac{2}{\pi}$-approximation by Nesterov [Nes98].

NP-hard to improve approximation constant. [BRS15]

[BRS15] Briët, Regev, and Saket. "Tight hardness of the non-commutative Grothendieck problem". 2015.

[Nes98] Nesterov. "Semidefinite relaxation and nonconvex quadratic optimization". 1998.

[GW95] Goemans and Williamson. "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming". 1995.

Max Cut

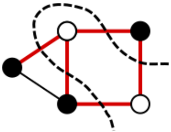
Given $A \succ 0$, the $\textsc{MaxCut}$ problem is:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle$$

For PSD A, we have a $\frac{2}{\pi}$-approximation by Nesterov [Nes98].

NP-hard to improve approximation constant. [BRS15]

Can be improved to 0.878 [GW95] if A is graph Laplacian.



[BRS15] Briët, Regev, and Saket. "Tight hardness of the non-commutative Grothendieck problem". 2015.

[Nes98] Nesterov. "Semidefinite relaxation and nonconvex quadratic optimization". 1998.

[GW95] Goemans and Williamson. "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming". 1995.

Max Cut as Optimization on the Sphere

An equivalent formulation over the sphere:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle = \max_{\|z\|_2 = 1} \left\| A^{\frac{1}{2}} z \right\|_1^2 .$$

Max Cut as Optimization on the Sphere

An equivalent formulation over the sphere:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle = \max_{\|z\|_2 = 1} \left\| A^{\frac{1}{2}} z \right\|_1^2 .$$

Using variational characterization $\langle x, Ax \rangle = \max_{y \in \mathbb{R}^n} 2 \langle x, y \rangle - \langle y, A^{-1} y \rangle$:

$$\max_{x \in \{\pm 1\}^n, y \in \mathbb{R}^n} 2 \langle x, y \rangle - \left\langle y, A^{-1} y \right\rangle = \max_{y \in \mathbb{R}^n} 2 \|y\|_1 - \left\langle y, A^{-1} y \right\rangle$$

Substitute $y = \lambda A^{\frac{1}{2}} z$, where $\|z\| = 1$, $\lambda \in \mathbb{R}$.

$$\max_{\|z\|_2 = 1, \lambda \in \mathbb{R}} 2\lambda \left\| A^{\frac{1}{2}} z \right\|_1 - \lambda^2 = \max_{\|z\|_2 = 1} \left\| A^{\frac{1}{2}} z \right\|_1^2 .$$

Max Cut as Optimization on the Sphere

An equivalent formulation over the sphere:

$$\max_{x \in \{\pm 1\}^n} \langle x, Ax \rangle = \max_{\|z\|_2 = 1} \left\| A^{\frac{1}{2}} z \right\|_1^2 .$$

Using variational characterization $\langle x, Ax \rangle = \max_{y \in \mathbb{R}^n} 2 \langle x, y \rangle - \langle y, A^{-1} y \rangle$:

$$\max_{x \in \{\pm 1\}^n, y \in \mathbb{R}^n} 2 \langle x, y \rangle - \left\langle y, A^{-1} y \right\rangle = \max_{y \in \mathbb{R}^n} 2 \|y\|_1 - \left\langle y, A^{-1} y \right\rangle$$

Substitute $y = \lambda A^{\frac{1}{2}} z$, where $\|z\| = 1$, $\lambda \in \mathbb{R}$.

$$\max_{\|z\|_2 = 1, \lambda \in \mathbb{R}} 2\lambda \left\| A^{\frac{1}{2}} z \right\|_1 - \lambda^2 = \max_{\|z\|_2 = 1} \left\| A^{\frac{1}{2}} z \right\|_1^2 .$$

Generalized to matrix $2 \to q$ **norms when** $0 < q <= 2$ ($\textsc{MaxCut}$ equivalent to $2 \to 1$ norm).

Permanent of PSD Matrices

Permanent of a matrix $\text{per}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$

#P-hard to compute in general, NP-hard to compute sign

[JSV04] Jerrum, Sinclair, and Vigoda. "A Polynomial-Time Approximation Algorithm for the Permanent of a Matrix with Nonnegative Entries". 2004.

[LSW] Linial, Samorodnitsky, and Wigderson. "A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents".

[Ana+17] Anari et al. "Simply Exponential Approximation of the Permanent of Positive Semidefinite Matrices". 2017.

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.

[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

Permanent of PSD Matrices

Permanent of a matrix $\text{per}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$

#P-hard to compute in general, NP-hard to compute sign

- $A \geq 0$: Counts bipartite perfect matchings
- $A \succeq 0$: Output probabilities of boson sampling experiment

[JSV04] Jerrum, Sinclair, and Vigoda. "A Polynomial-Time Approximation Algorithm for the Permanent of a Matrix with Nonnegative Entries". 2004.

[LSW] Linial, Samorodnitsky, and Wigderson. "A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents".

[Ana+17] Anari et al. "Simply Exponential Approximation of the Permanent of Positive Semidefinite Matrices". 2017.

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.

[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

Permanent of PSD Matrices

Permanent of a matrix $\mathrm{per}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$

#P-hard to compute in general, NP-hard to compute sign

- $A \geq 0$: Counts bipartite perfect matchings
- $A \succeq 0$: Output probabilities of boson sampling experiment

For $A \geq 0$, $(1+\epsilon)$ randomized approximation algorithm [JSV04], e^{-n} deterministic approximation algorithm [LSW].

[JSV04] Jerrum, Sinclair, and Vigoda. "A Polynomial-Time Approximation Algorithm for the Permanent of a Matrix with Nonnegative Entries". 2004.
[LSW] Linial, Samorodnitsky, and Wigderson. "A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents".
[Ana+17] Anari et al. "Simply Exponential Approximation of the Permanent of Positive Semidefinite Matrices". 2017.
[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.
[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

Permanent of PSD Matrices

Permanent of a matrix $\text{per}(A) = \sum_\sigma \prod_{i=1}^n A_{i,\sigma(i)}$

#P-hard to compute in general, NP-hard to compute sign

- $A \geq 0$: Counts bipartite perfect matchings
- $A \succeq 0$: Output probabilities of boson sampling experiment

For $A \geq 0$, $(1+\epsilon)$ randomized approximation algorithm [JSV04], e^{-n} deterministic approximation algorithm [LSW].

For $A \succeq 0$, $e^{-(1+\gamma)n}$ approximation algorithm [Ana+17; YP21].

NP-hard to approximate better than $e^{-\gamma n}$, current-best approximation factor $e^{-(0.9999+\gamma)n}$ [ENOG24].

[JSV04] Jerrum, Sinclair, and Vigoda. "A Polynomial-Time Approximation Algorithm for the Permanent of a Matrix with Nonnegative Entries". 2004.

[LSW] Linial, Samorodnitsky, and Wigderson. "A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents".

[Ana+17] Anari et al. "Simply Exponential Approximation of the Permanent of Positive Semidefinite Matrices". 2017.

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.

[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

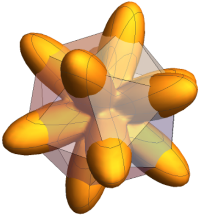
Approximating Permanents of PSD Matrices

Let $A = V^\dagger V$ and v_i be the columns of V

We study the following polynomial optimization problem on the complex $(n-1)$-sphere of radius $\sqrt{n}$, and its SDP relaxation:

$$\rho(A) \equiv \max_{\|x\|^2=n} \prod_{i=1}^{n} |\langle x, v_i \rangle|^2$$

$$\mathrm{rel}(A) \equiv \max_{X} \prod_{i=1}^{n} \left\langle X, v_i v_i^\dagger \right\rangle \quad \text{s.t.} \begin{cases} \mathrm{Tr}(X) = n \\ X \succeq 0 \end{cases}$$



Product of Linear Forms and Permanent Approximation

Theorem ([YP21])

Let $\gamma = 0.5772...$ be the Euler-Mascheroni constant. Then

$$e^{-\gamma n}\text{rel}(A) \overset{1}{\leq} \rho(A) \leq \text{rel}(A)$$

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.

[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

Product of Linear Forms and Permanent Approximation

Theorem ([YP21])

Let $\gamma = 0.5772\ldots$ *be the Euler-Mascheroni constant. Then*

$$e^{-\gamma n}\mathrm{rel}(A) \overset{1}{\leq} \rho(A) \leq \mathrm{rel}(A)$$

Connection to permanent:

Theorem ([YP21])

$$e^{-n(1+\gamma)}\mathrm{rel}(A) \overset{1}{\leq} e^{-n}\rho(A) \leq \mathrm{per}(A) \overset{2}{\leq} \mathrm{rel}(A)$$

Approximating permanent $\longleftrightarrow$ polynomial optimization over sphere

---

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.

[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

Product of Linear Forms and Permanent Approximation

Theorem ([YP21])

Let $\gamma = 0.5772...$ be the Euler-Mascheroni constant. Then

$$e^{-\gamma n}\mathrm{rel}(A) \overset{1}{\leq} \rho(A) \leq \mathrm{rel}(A)$$

Connection to permanent:

Theorem ([YP21])

$$e^{-n(1+\gamma)}\mathrm{rel}(A) \overset{1}{\leq} e^{-n}\rho(A) \leq \mathrm{per}(A) \overset{2}{\leq} \mathrm{rel}(A)$$

Approximating permanent $\longleftrightarrow$ polynomial optimization over sphere

(1) is tight for low-rank matrices, (2) is tight for high-rank matrices, [ENOG24] found arbitrage to improve approximation by 10^{-4}.

[YP21] Yuan and Parrilo. "Maximizing Products of Linear Forms, and the Permanent of Positive Semidefinite Matrices". 2021.
[ENOG24] Ebrahimnejad, Nagda, and Oveis Gharan. "On approximability of the Permanent of PSD matrices". 2024.

A unified perspective

Given $A_1, \ldots, A_d \succeq 0$. Consider the image of the sphere $\{x \in \mathbb{K}^n \mid \|x\| = 1\}$ under the map

$$\mathcal{A} : \mathbb{K}^n \to \mathbb{R}^d_+$$
$$\mathcal{A}(x) = (\langle x, A_1 x\rangle, \ldots, \langle x, A_d x\rangle)$$

A unified perspective

Given $A_1, \ldots, A_d \succeq 0$. Consider the image of the sphere $\{x \in \mathbb{K}^n \mid \|x\| = 1\}$ under the map

$$\mathcal{A} : \mathbb{K}^n \to \mathbb{R}^d_+$$
$$\mathcal{A}(x) = (\langle x, A_1 x\rangle, \ldots, \langle x, A_d x\rangle)$$

$\mathrm{Im}(\mathcal{A})$ is a non-convex set in $\mathbb{R}^d$, convex hull given by

$$\mathrm{conv}(\mathrm{Im}(\mathcal{A})) = \{(\langle X, A_1\rangle, \ldots, \langle X, A_d\rangle) \mid \mathrm{Tr}(X) = 1, X \succeq 0\}$$

A unified perspective

Given $A_1, \ldots, A_d \succeq 0$. Consider the image of the sphere $\{x \in \mathbb{K}^n \mid \|x\| = 1\}$ under the map

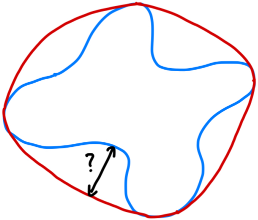
$$\mathcal{A} : \mathbb{K}^n \to \mathbb{R}^d_+$$
$$\mathcal{A}(x) = (\langle x, A_1 x \rangle, \ldots, \langle x, A_d x \rangle)$$

$\mathrm{Im}(\mathcal{A})$ is a non-convex set in $\mathbb{R}^d$, convex hull given by

$$\mathrm{conv}(\mathrm{Im}(\mathcal{A})) = \{(\langle X, A_1 \rangle, \ldots, \langle X, A_d \rangle) \mid \mathrm{Tr}(X) = 1, X \succeq 0\}$$

How well does $\mathrm{conv}(\mathrm{Im}(\mathcal{A}))$ approximate $\mathrm{Im}(\mathcal{A})$?

What measure of approximation to use?



A unified perspective

We study the following relaxation

$$\max_{\|x\|=1} \sum_i f(\langle x, A_i x \rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X \succeq 0} \sum_i f(\langle X, A_i \rangle),$$

where $A_i \succeq 0$ and f is concave

A unified perspective

We study the following relaxation

$$\max_{\|x\|=1} \sum_i f(\langle x, A_i x\rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X\succeq 0} \sum_i f(\langle X, A_i\rangle),$$

where $A_i \succeq 0$ and f is concave

$f(a) = a^p$ for $0 \le p \le 1$: Matrix $2 \to \frac{p}{2}$ norm:

- $f(a) = \sqrt{a}$: MaxCut with PSD objective (ℓ_1 norm)
- $f(a) = \log a$: Permanents and product of PSD forms (KL-divergence)

A unified perspective

We study the following relaxation

$$\max_{\|x\|=1} \sum_i f(\langle x, A_i x\rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X\succeq 0} \sum_i f(\langle X, A_i\rangle),$$

where $A_i \succeq 0$ and f is concave

$f(a) = a^p$ for $0 \leq p \leq 1$: Matrix $2 \to \frac{p}{2}$ norm:

- $f(a) = \sqrt{a}$: MaxCut with PSD objective (ℓ_1 norm)
- $f(a) = \log a$: Permanents and product of PSD forms (KL-divergence)

$f(a) = -a^p$ for $p \geq 1$: Subspace approximation for $d = 1$

A unified perspective

We study the following relaxation

$$\max_{\|x\|=1} \sum_i f(\langle x, A_i x\rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X\succeq 0} \sum_i f(\langle X, A_i\rangle),$$

where $A_i \succeq 0$ and f is concave

$f(a) = a^p$ for $0 \leq p \leq 1$: Matrix $2 \to \frac{p}{2}$ norm:

- $f(a) = \sqrt{a}$: $\textsc{MaxCut}$ with PSD objective (ℓ_1 norm)
- $f(a) = \log a$: Permanents and product of PSD forms (KL-divergence)

$f(a) = -a^p$ for $p \geq 1$: Subspace approximation for $d = 1$

$f(a) = -a \log a$: Entropy maximization of POVM (Reverse KL)

Summary of results

Rounding algorithm: Given optimum $X^* = UU^\dagger$, produce unit vector $\hat{x}$ by:

- Sample $x \sim N(0, X^*)$
- Normalize $\hat{x} = x / \|x\|$

Summary of results

Rounding algorithm: Given optimum $X^* = UU^\dagger$, produce unit vector $\hat{x}$ by:

- Sample $x \sim N(0, X^*)$
- Normalize $\hat{x} = x / \|x\|$

$f(a)$	Approximation	Factor ($\mathbb{K} = \mathbb{R}$)	Factor ($\mathbb{K} = \mathbb{C}$)
a^p	Multiplicative	$2^p \Gamma(1/2 + p) / \sqrt{\pi}$	$\Gamma(1 + p)$
$\log a$	Additive	$\gamma + \log 2$	γ
$-a \log a$	Additive	2	1

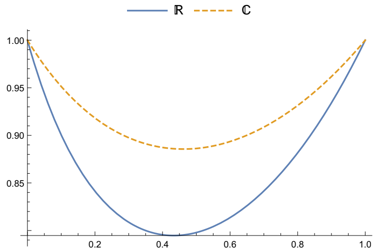
Additional gains when we can bound the rank of SDP solution X^*

Improved bounds for low-rank solutions

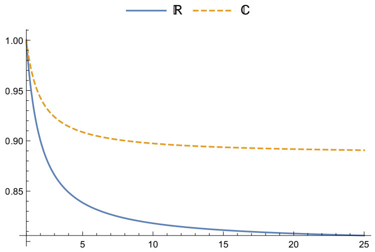
When $f(a) = a^p$ and the solution of X^* has rank r, approximation factor

$$\alpha_f(\mathbb{R}) = \frac{r^p\Gamma(r/2)\Gamma(1/2+p)}{\Gamma(1/2)\Gamma(r/2+p)} > \frac{2^p\Gamma(1/2+p)}{\Gamma(1/2)}$$

$$\alpha_f(\mathbb{C}) = \frac{r^p\Gamma(r)\Gamma(1+p)}{\Gamma(r+p)} > \Gamma(1+p)$$



(a) $\alpha_f(\mathbb{K})$ against p when $r \to \infty$



(b) $\alpha_f(\mathbb{K})$ against r when $p = \frac{1}{2}$

Rounding Analysis

$$\max_{\|x\|=1} \sum_i \log(\langle x, A_i x \rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X \succeq 0} \sum_i \log(\langle X, A_i \rangle),$$

Rounding Analysis

$$\max_{\|x\|=1} \sum_i \log(\langle x, A_i x\rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X\succeq 0} \sum_i \log(\langle X, A_i\rangle),$$

Rounding algorithm: Given rank-r optimum $X^* = UU^\dagger$, produce $\hat{x}$:

- Sample $x \sim N(0, X^*)$
- Normalize $\hat{x} = x / \|x\|$

Rounding Analysis

$$\max_{\|x\|=1} \sum_i \log(\langle x, A_i x \rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X \succeq 0} \sum_i \log(\langle X, A_i \rangle),$$

Rounding algorithm: Given rank-r optimum $X^* = UU^\dagger$, produce $\hat{x}$:

- Sample $x \sim N(0, X^*)$
- Normalize $\hat{x} = x / \|x\|$

Analyze:

$$\mathbb{E}\left[\sum_i \log(\langle \hat{x}, A_i \hat{x} \rangle)\right] = \sum_i \mathop{\mathbb{E}}_{z \sim N(0,I)} \left[\log z^\dagger U^\dagger A_i U z - \log z^\dagger U^\dagger U z\right]$$

Rounding Analysis

$$\max_{\|x\|=1} \sum_i \log(\langle x, A_i x\rangle) \quad \xrightarrow{\text{relax}} \quad \max_{\mathrm{Tr}(X)=1,\, X\succeq 0} \sum_i \log(\langle X, A_i\rangle),$$

Rounding algorithm: Given rank-r optimum $X^* = UU^\dagger$, produce $\hat{x}$:

- Sample $x \sim N(0, X^*)$
- Normalize $\hat{x} = x/\|x\|$

Analyze:

$$\mathbb{E}\left[\sum_i \log(\langle \hat{x}, A_i \hat{x}\rangle)\right] = \sum_i \mathop{\mathbb{E}}_{z\sim N(0,I)}\left[\log z^\dagger U^\dagger A_i U z - \log z^\dagger U^\dagger U z\right]$$

Key Step: $g(\lambda) = \mathbb{E}\log\sum_{i=1}^r \lambda_i |z_i|^2$ is a symmetric concave function

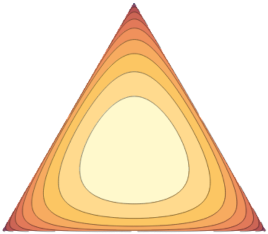
$$\mathbb{E}\log z^\dagger U^\dagger U z = \mathbb{E}\log\left(\sum_{i=1}^r \lambda_i |z_i|^2\right) \le \mathbb{E}\log\left(\frac{1}{r}\sum_{i=1}^r |z_i|^2\right) = H_{r-1} - \gamma - \log(r)$$

Main Technical Tool/Trick

Bounding function on a domain with symmetry, e.g. simplex
$\Delta_r = \{\lambda \mid \lambda \geq 0, \sum_i \lambda_i = 1\}$

Maximum (minimum) of symmetric concave (convex) function on the simplex achieved at center: $\lambda_i = \frac{1}{r}$.

Minumum (maximum) of symmetric concave (convex) function on the simplex achieved at vertices: $\lambda_1 = 1, \lambda_2, \ldots, \lambda_r = 0$.



Conclusion

Summary:

Simple technique for obtaining tight bounds on expected values of concave functions of Gaussian variables.

Tight analysis (NP-hard to improve) for Max-Cut, product of linear forms, subspace approximation.

Conclusion

Summary:

Simple technique for obtaining tight bounds on expected values of concave functions of Gaussian variables.

Tight analysis (NP-hard to improve) for Max-Cut, product of linear forms, subspace approximation.

Questions:

Why do these tight approximation bounds depend on properties of Gaussian variables, though they are not mentioned in the problem description?

Are there other classes of functions of Gaussian random variables for which we can obtain similarly tight approximation bounds?